7 Entanglement

In this chapter we revisit in a more formal manner the phenomenon of quantum mechanical entanglement, which we briefly encountered in Chapter 2. We will consider the case of polarization-entangled photons and study such systems with the goal to discriminate between the quantum mechanics vs. classical "hidden variables" formalisms. The Bell inequalities will be presented and shown to provide a setting for experimental tests that provide a verdict on the physical reality underlying such systems.

Some of the material presented in this chapter is taken from Auletta, Fortunato and Parisi, Chap. 16, and Grynberg, Aspect and Fabre, Chap. 5.

7.1 Polarization-entangled Photons

We have studied at the end of Chapter 2 an example of two spin-1/2 particles whose states (i.e., "up" or "down" along a given orientation) were entangled. That is, their combined, compound state could not be expressed as a direct product of states defined independently in the particles respective spaces. It was then found that the measurement of the spin state in one particle completely determined that of the other (see Example 2.4). We now revisit this problem but within the context of pairs of polarization-entangled photons.

7.1.1 Polarization States and Polarizers

Before we tackle the problem of polarization-entangled pairs of photons, let us define the parameters at play by considering two modes p and p' available to a single photon; these modes are as defined in Chapter 6. That is, given the wave vector $\mathbf{k}$ determining the direction of propagation shared by the two modes, p and p' respectively have $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}'$ polarization states located in the plane perpendicular to $\mathbf{k}$, as well as being perpendicular to one another. We therefore have

$$\boldsymbol{\varepsilon} \cdot \mathbf{k} = 0 \tag{7.1}$$

$$\boldsymbol{\varepsilon}' \cdot \mathbf{k} = 0 \tag{7.2}$$

$$\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}' = 0 \tag{7.3}$$

and, say,

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}' \times \frac{\mathbf{k}}{k}. \tag{7.4}$$

We introduce a device, a *polarizer*, that can measure or identify the polarization state of the photon. For example, if we consider a photon moving along the z-axis we can use the basis $\{|x\rangle, |y\rangle\}$ to characterize the polarization states along the x- and y-axes, then the polarizer can be oriented to transmit the photon towards two output ports labeled $+1$ and -1 depending on whether the polarization state is $|x\rangle$ or $|y\rangle$, respectively. This will be the case when one of the polarizer's two principal axes is aligned with, say, the x-axis. If we assign the operator $\hat{A}(0)$ to the polarizer in this specific arrangement, this is equivalent to saying that the kets $|x\rangle$ or $|y\rangle$ have eigenvalues of $+1$ and -1 when a polarization measurement is performed. That is,

$$\begin{aligned} \hat{A}(0)|x\rangle &= |x\rangle & (7.5) \\ \hat{A}(0)|y\rangle &= -|y\rangle. & (7.6) \end{aligned}$$

Of course, the choice of the basis $\{|x\rangle, |y\rangle\}$ is arbitrary and, more generally, we could use two perpendicular axes rotated by an angle θ relative to the x and y-axes to define the polarization states (and basis)

$$\begin{aligned} |+\theta\rangle &= \cos(\theta)|x\rangle + \sin(\theta)|y\rangle & (7.7) \\ |-\theta\rangle &= -\sin(\theta)|x\rangle + \cos(\theta)|y\rangle. & (7.8) \end{aligned}$$

It follows that a polarizer with its principal axes also rotated by θ relative to the x and y-axes will verify the relations

$$\begin{aligned} \hat{A}(\theta)|+\theta\rangle &= |+\theta\rangle & (7.9) \\ \hat{A}(\theta)|-\theta\rangle &= -|-\theta\rangle. & (7.10) \end{aligned}$$

If we define the $\{|x\rangle, |y\rangle\}$ basis with

$$|x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{7.11}$$

then it is easily verified that equations (7.9)-(7.10) result when

$$\hat{A}(\theta) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}. \tag{7.12}$$

Evidently, we could also use the projectors $|+\theta\rangle\langle+\theta|$ and $|-\theta\rangle\langle-\theta|$ associated with the two corresponding subspaces to write

$$\hat{A}(\theta) = |+\theta\rangle\langle+\theta| - |-\theta\rangle\langle-\theta|. \tag{7.13}$$

Keeping the polarizer's principal axes in the same θ-orientation, we now consider a photon in the polarization state

$$|+\lambda\rangle = \cos(\lambda)|x\rangle + \sin(\lambda)|y\rangle \tag{7.14}$$

and calculate the probability of measuring the $|\pm\theta\rangle$ polarization states probed by the polarizer. We therefore write

$$\begin{aligned} |+\lambda\rangle &= (|+\theta\rangle\langle+\theta| + |-\theta\rangle\langle-\theta|)|+\lambda\rangle \\ &= \langle+\theta\,|+\lambda\rangle|+\theta\rangle + \langle-\theta\,|+\lambda\rangle|-\theta\rangle \\ &= [\cos(\theta)\cos(\lambda) + \sin(\theta)\sin(\lambda)]|+\theta\rangle \\ &\qquad + [-\sin(\theta)\cos(\lambda) + \cos(\theta)\sin(\lambda)]|-\theta\rangle \\ &= \cos(\lambda-\theta)|+\theta\rangle + \sin(\lambda-\theta)|-\theta\rangle \end{aligned} \tag{7.15}$$

and

$$\begin{aligned} \mathcal{P}(+\theta,+\lambda) &= |\langle+\theta\,|+\lambda\rangle|^2 \\ &= \cos^2(\lambda-\theta) \end{aligned} \tag{7.16}$$

$$\begin{aligned} \mathcal{P}(-\theta,+\lambda) &= |\langle-\theta\,|+\lambda\rangle|^2 \\ &= \sin^2(\lambda-\theta). \end{aligned} \tag{7.17}$$

These relations give the most general results for the probabilities of measuring two orthogonal polarization states for a photon polarized at an arbitrary angle to the polarizer used for the measurements (i.e., at angle $\lambda-\theta$ in the previous example).

7.1.2 Photon Pairs

We now consider two photons propagating in opposing directions; we choose the positive and negative z-axes. These two photons form a compound system and the set $\{|x_1,x_2\rangle, |x_1,y_2\rangle, |y_1,x_2\rangle, |y_1,y_2\rangle\}$ forms a basis that can be used to express any combination of their polarization states. We assume that two polarizers are positioned along the paths of the photons, one for each photon. The polarizer associated to the first photon has principal axes that make an angle θ_1 with the x- and y-axes, while the other (for measurements on the second photon) is rotated by θ_2.

Let us consider the combined polarization state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|x_1,x_2\rangle + |y_1,y_2\rangle), \tag{7.18}$$

which is readily determined to entangled. We can now inquire about the joint probabilities of finding the photons in either polarization states. Proceeding in a manner similar as for a single photon we use the projector for the compound space to express the system's polarization state

$$
\begin{aligned}
|\psi\rangle &= \left(\sum_{\alpha,\beta=+,-} |\alpha\theta_1,\beta\theta_2\rangle\langle\alpha\theta_1,\beta\theta_2|\right)|\psi\rangle \\
&= \frac{1}{\sqrt{2}}\left\{\left[\cos(\theta_1)\cos(\theta_2)+\sin(\theta_1)\sin(\theta_2)\right]\left(|+\theta_1,+\theta_2\rangle+|-\theta_1,-\theta_2\rangle\right)\right. \\
&\qquad \left.-\left[\sin(\theta_2)\cos(\theta_1)-\sin(\theta_1)\cos(\theta_2)\right]\left(|+\theta_1,-\theta_2\rangle-|-\theta_1,+\theta_2\rangle\right)\right\} \\
&= \frac{1}{\sqrt{2}}\left[\cos(\theta_1-\theta_2)\left(|+\theta_1,+\theta_2\rangle+|-\theta_1,-\theta_2\rangle\right)\right. \\
&\qquad \left.+\sin(\theta_1-\theta_2)\left(|+\theta_1,-\theta_2\rangle-|-\theta_1,+\theta_2\rangle\right)\right], \qquad (7.19)
\end{aligned}
$$

and for the different probabilities

$$
\begin{aligned}
\mathcal{P}(+\theta_1,+\theta_2) &= \mathcal{P}(-\theta_1,-\theta_2) \\
&\quad \left|\langle\pm\theta_1,\pm\theta_2\,|\,\psi\rangle\right|^2 \\
&= \frac{1}{2}\cos^2(\theta_1-\theta_2) \qquad (7.20) \\
\mathcal{P}(+\theta_1,-\theta_2) &= \mathcal{P}(-\theta_1,+\theta_2) \\
&= \left|\langle\pm\theta_1,\mp\theta_2\,|\,\psi\rangle\right|^2 \\
&= \frac{1}{2}\sin^2(\theta_1-\theta_2). \qquad (7.21)
\end{aligned}
$$

These equations show that the polarization states of the photons are correlated. We can also calculate the probabilities for the polarization state of one photon with regard to the state of the other

$$
\begin{aligned}
\mathcal{P}(+\theta_1) &= \mathcal{P}(+\theta_1,+\theta_2)+\mathcal{P}(+\theta_1,-\theta_2) \\
&= \frac{1}{2} \qquad (7.22) \\
\mathcal{P}(-\theta_1) &= \mathcal{P}(-\theta_1,+\theta_2)+\mathcal{P}(-\theta_1,-\theta_2) \\
&= \frac{1}{2}, \qquad (7.23)
\end{aligned}
$$

with similar results if we reverse the roles of the photons (i.e., $\mathcal{P}(\pm\theta_2)=\mathcal{P}(\pm\theta_1)$).

We note that if the polarization state of the first photon is measured to be $|+\theta_1\rangle$ then the state of the system afterwards is given by (see equations (2.87) and (2.91) in Chapter 2)

$$
|\psi'\rangle = \frac{\left(\sum_{\beta=+,-}|+\theta_1,\beta\theta_2\rangle\langle+\theta_1,\beta\theta_2|\right)|\psi\rangle}{\sqrt{\sum_{\beta=+,-}\left|\langle+\theta_1,\beta\theta_2\,|\,\psi\rangle\right|^2}}
$$

$$
\begin{aligned}
&= \frac{\frac{1}{\sqrt{2}}\left[\cos\left(\theta_1-\theta_2\right)\left|+\theta_1,+\theta_2\right\rangle+\sin\left(\theta_1-\theta_2\right)\left|+\theta_1,-\theta_2\right\rangle\right]}{\sqrt{\frac{1}{2}\left[\cos^2\left(\theta_1-\theta_2\right)+\sin^2\left(\theta_1-\theta_2\right)\right]}}\\
&= \left|+\theta_1\right\rangle\otimes\left[\cos\left(\theta_1-\theta_2\right)\left|+\theta_2\right\rangle+\sin\left(\theta_1-\theta_2\right)\left|-\theta_2\right\rangle\right]. \qquad (7.24)
\end{aligned}
$$

As we saw in Chapter 2, the system is no more entangled after the measurement since it can now be expressed as a direct product. Furthermore, comparison with equations (7.14)-(7.15) shows that the polarization of the second photon after the measurement can be written as

$$
\begin{aligned}
\left|\psi_2'\right\rangle &= \cos\left(\theta_1-\theta_2\right)\left|+\theta_2\right\rangle+\sin\left(\theta_1-\theta_2\right)\left|-\theta_2\right\rangle\\
&= \left|+\theta_1\right\rangle. \qquad (7.25)
\end{aligned}
$$

That is, the polarization state is found to be the same as that of the first photon. The measurement on the first photon has therefore set the polarization state of the second. This reveals a very high degree of correlation between their polarization states.

This correlation can be ascertained in a different manner, which will reveal itself to be useful later on. We assign to the measurement outcome on the first photon a random variable $\mathcal{A}(\theta_1)$ that can (randomly) take the ± 1 values only. Similarly, we ascribe another such variable $\mathcal{B}(\theta_2)$ for the measurement on the second photon. Because of their random nature we have $E\{\mathcal{A}(\theta_1)\}=E\{\mathcal{B}(\theta_2)\}=0$ for the mean values ($E\{X\}$ stands for "the expected value of X"). Furthermore, the variances are

$$
\begin{aligned}
E\left\{\mathcal{A}^2\left(\theta_1\right)\right\} &= E\left\{\mathcal{B}^2\left(\theta_2\right)\right\}\\
&= \frac{1}{2}\left[1^2+(-1)^2\right]\\
&= 1, \qquad (7.26)
\end{aligned}
$$

and the correlation coefficient between the two random variables is defined with

$$
\begin{aligned}
\operatorname{Corr}\left[\mathcal{A}\left(\theta_1\right)\mathcal{B}\left(\theta_2\right)\right] &\equiv \frac{E\left\{\mathcal{A}\left(\theta_1\right)\mathcal{B}\left(\theta_2\right)\right\}-E\left\{\mathcal{A}\left(\theta_1\right)\right\}E\left\{\mathcal{B}\left(\theta_2\right)\right\}}{\sqrt{E\left\{\mathcal{A}^2\left(\theta_1\right)\right\}}\sqrt{E\left\{\mathcal{B}^2\left(\theta_2\right)\right\}}}\\
&= E\left\{\mathcal{A}\left(\theta_1\right)\mathcal{B}\left(\theta_2\right)\right\}. \qquad (7.27)
\end{aligned}
$$

From the possible outcomes calculated in equations (7.20)-(7.21) we find that, according to quantum mechanics, the correlation between the polarization states of the photons is

$$
\begin{aligned}
E_{\mathrm{QM}}\left\{\mathcal{A}\left(\theta_1\right)\mathcal{B}\left(\theta_2\right)\right\} &= \mathcal{P}\left(+\theta_1,+\theta_2\right)+\mathcal{P}\left(-\theta_1,-\theta_2\right)-\mathcal{P}\left(+\theta_1,-\theta_2\right)-\mathcal{P}\left(-\theta_1,+\theta_2\right)\\
&= \cos^2\left(\theta_1-\theta_2\right)-\sin^2\left(\theta_1-\theta_2\right)\\
&= \cos\left[2\left(\theta_1-\theta_2\right)\right]. \qquad (7.28)
\end{aligned}
$$

We can also calculate the conditional probability of finding the second photon in the $|\psi_2'\rangle = |+\theta_1\rangle$ knowing that the first photon was previously measured to be in the $|+\theta_1\rangle$. This yields

$$\begin{aligned}\mathcal{P}\left[\mathcal{B}\left(\theta_2 = +\theta_1\right)|\mathcal{A}\left(+\theta_1\right)\right] &= \frac{\mathcal{P}\left[\mathcal{B}\left(\theta_2 = +\theta_1\right) \cap \mathcal{A}\left(+\theta_1\right)\right]}{\mathcal{P}\left[\mathcal{A}\left(+\theta_1\right)\right]} \\ &= \frac{\mathcal{P}\left(+\theta_1, +\theta_1\right)}{\mathcal{P}\left(+\theta_1\right)} \\ &= 1. \end{aligned} \tag{7.29}$$

We once again find the aforementioned correlation between the polarization states of the photons, i.e., the measurement on the first photon completely determines the state of the second photon.

7.1.3 A Classical View - Hidden Variables

It is interesting to note that it is Einstein who first realized the predictions of entanglement made by quantum mechanics. Since for the case of the photon pairs we discussed above the distance between the two polarizers could in principle by very large, the quantum mechanical predictions may seem to imply the transmission of information between the two polarization measurement sites at speeds that could exceed the speed of light (if one adopts a point of view based on classical mechanics notions). It is therefore not surprising that Einstein, who was the father of relativity, would be at odds with such predictions, which he interpreted as clear indications of the shortcomings of quantum mechanics. He therefore endeavoured to find alternative explanations to account for (potential) experiments that may appear to corroborate these quantum mechanical predictions. This was the approach taken in the famous Einstein-Podolsky-Rosen (EPR) paradox paper[1].

To make clear the ideas to be considered in this section we go back to our previous calculations and set $\theta_1 = \theta_2 = 0$ in equations (7.20)-(7.21) such that both polarizers are aligned with the x-axis. We therefore find that, as one would reasonably expect, that a set of measurements would reveal equal probabilities $\mathcal{P}\left(+\theta_1, +\theta_2\right) = \mathcal{P}\left(-\theta_1, -\theta_2\right) = 1/2$ of finding the photons polarized along the x- and y-axes, while $\mathcal{P}\left(+\theta_1, -\theta_2\right) = \mathcal{P}\left(-\theta_1, +\theta_2\right) = 0$ for having oppositely polarized photon pairs. One could imagine a simple (or simplistic) non-quantum mechanical scheme to explain such results. For example, if we could produce a source of photons that simultaneous output pairs of photons of which half are polarized along the x-axis and the other along the y-axis, a series of measurements would recover the same probabilities. It is important to note that this (classical) realization of the results needed the addition of an additional assumption, i.e., the presence of two different polarization states for the photon pairs. This is different from our quantum mechanical analysis, which assumes the same state

[1]Einstein, A., Podolsky, B., Rosen, N. 1935, "Can Quantum-Mechanical Description of Physical Reality be Considered Complete?" Physical Review, 47 (10): 777–780.

for all photons (see equation (7.18)). The adoption of a scheme that requires additional assumptions or quantities is equivalent to a statement that quantum mechanical is incomplete and requires to be augmented in some fashion. Such quantities are called ***hidden variables***, and are at the heart of the EPR paradox.

Evidently, this simple scheme would fail in more general situations where, for example, $\theta_1, \theta_2 \neq 0$, but it could in principle be refined or adapted to these more complicated set-ups. For example, we could generalize the possible angle (relative to the x-axis) characterizing the polarization state of any pair of photon to randomly take any value between 0 and 2π. That is, the probability density function for this angle λ is given by[2]

$$\rho(\lambda) = \frac{1}{2\pi}. \tag{7.30}$$

We also model and idealize a polarizer in such a manner that they yield a measurement of $+1$ or -1 depending whether the polarization state of the photon is closest to one of its principal axes. The following functions for the two polarizers verify this condition

$$\begin{aligned}
\mathcal{A}(\lambda, \theta_1) &= \operatorname{sign}\{\cos[2(\theta_1 - \lambda)]\} && (7.31)\\
\mathcal{B}(\lambda, \theta_2) &= \operatorname{sign}\{\cos[2(\theta_2 - \lambda)]\}, && (7.32)
\end{aligned}$$

where $\operatorname{sign}\{x\} = +1$ or -1 according to whether $x > 0$ or $x < 0$, respectively. We can see that a photon will register a measurement of $+1$ when $|\theta_1 - \lambda| < \pi/4$ and -1 otherwise.

The correlation between a pair of measurements (one on each polarimeter) yields

$$\begin{aligned}
E_{\mathrm{HV}}\{\mathcal{A}(\theta_1)\mathcal{B}(\theta_2)\} &= \frac{1}{2\pi}\int_0^{2\pi} d\lambda\rho(\lambda)\mathcal{A}(\lambda, \theta_1)\mathcal{B}(\lambda, \theta_2)\\
&= \frac{1}{2\pi}\int_0^{2\pi} d\lambda\rho(\lambda)\operatorname{sign}\{\cos[2(\theta_1 - \lambda)]\cos[2(\theta_2 - \lambda)]\}\\
&= \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\tau\rho(\tau)\operatorname{sign}\{\cos[2(\theta_1 - \theta_2) - \tau]\cos(-\tau)\}, \qquad (7.33)
\end{aligned}$$

where we used $\tau = 2(\lambda - \theta_2)$ and recognized that the integral is invariant when performed on any range of angles that measures 2π. To evaluate this integral, let us assume for simplicity that $\Delta\theta \equiv \theta_1 - \theta_2 > 0$. When then find that $\operatorname{sign}\{\cos[2\Delta\theta - \tau]\cos(-\tau)\} = 1$ when $-\pi/2 < -\tau < \pi/2 - 2\Delta\theta$ (i.e., both cosine functions are positive) and $\pi/2 < -\tau < 3\pi/2 - 2\Delta\theta$ (i.e., both cosine functions are negative). It follows that the range over which the integrand is positive totals $2\pi - 4\Delta\theta$, while it is negative over a range of $4\Delta\theta$. It can easily be verified that a similar analysis for $\Delta\theta < 0$ gives the same result. We therefore find that

[2]This assumption of an evenly distributed polarization angle is the hidden variable in this example.

$$\begin{aligned} E_{\mathrm{HV}}\left\{\mathcal{A}\left(\theta_1\right)\mathcal{B}\left(\theta_2\right)\right\} &= \frac{1}{2\pi}\left[\left(2\pi-4\Delta\theta\right)-4\Delta\theta\right] \\ &= 1-\frac{4}{\pi}\left|\theta_1-\theta_2\right|. \end{aligned} \tag{7.34}$$

A close comparison of equations (7.28) and (7.34) reveals that this hidden variable and the quantum mechanical models are quite close to one another. For example, they both give correlations 0 for $\theta_1-\theta_2=\pi/4$, $+1$ when $\theta_1-\theta_2=0$ and -1 when $\theta_1-\theta_2=\pi/2$. Elsewhere the difference between the two functions is not very large. Since the hidden variable result was obtain with what is still a fairly crude representation for the polarizers, we may reasonably think that further refinements may reduce the differences with the quantum mechanical model to a small enough level to be undetectable. This question was eventually addressed through the so called ***Bell inequalities***, which we now introduce.

7.1.4 The Bell Inequalities

Let us now consider a classical model based on a hidden variable λ characterizing a pair of photons, but which changes randomly from one pair to another. We assume that the probability density of this hidden variable is positive definite

$$\rho\left(\lambda\right) \geq 0 \tag{7.35}$$

$$\int d\lambda\rho\left(\lambda\right) = 1 \tag{7.36}$$

and we use, as before, polarizers that can only yield measurements $\left|\mathcal{A}\left(\theta_1\right)\right|=\left|\mathcal{B}\left(\theta_2\right)\right|=\pm1$. We further expect that the measurements performed with the polarizers are zero-mean with

$$\begin{aligned} \int d\lambda\rho\left(\lambda\right)\mathcal{A}\left(\theta_1\right) &= \int d\lambda\rho\left(\lambda\right)\mathcal{B}\left(\theta_2\right) \\ &= 0. \end{aligned} \tag{7.37}$$

Also, the correlation between measurements on pairs of photons is given by

$$E_{\mathrm{HV}}\left\{\mathcal{A}\left(\theta_1\right)\mathcal{B}\left(\theta_2\right)\right\} = \int d\lambda\rho\left(\lambda\right)\mathcal{A}\left(\theta_1\right)\mathcal{B}\left(\theta_2\right). \tag{7.38}$$

We now select two angles for each polarizers, i.e., $\left(\theta_1,\theta_1'\right)$ and $\left(\theta_2,\theta_2'\right)$, and define the following quantity

$$\begin{aligned} s\left(\lambda,\theta_1,\theta_1',\theta_2,\theta_2'\right) &= \mathcal{A}\left(\theta_1\right)\mathcal{B}\left(\theta_2\right)-\mathcal{A}\left(\theta_1\right)\mathcal{B}\left(\theta_2'\right)+\mathcal{A}\left(\theta_1'\right)\mathcal{B}\left(\theta_2\right)+\mathcal{A}\left(\theta_1'\right)\mathcal{B}\left(\theta_2'\right) \\ &= \mathcal{A}\left(\theta_1\right)\left[\mathcal{B}\left(\theta_2\right)-\mathcal{B}\left(\theta_2'\right)\right]+\mathcal{A}\left(\theta_1'\right)\left[\mathcal{B}\left(\theta_2\right)+\mathcal{B}\left(\theta_2'\right)\right] \\ &= \pm2. \end{aligned} \tag{7.39}$$

The last step is due to our requirement that $|\mathcal{A}(\theta_1)| = |\mathcal{B}(\theta_2)| = \pm 1$, which implies that $\mathcal{B}(\theta_2) = \pm\mathcal{B}(\theta_2')$. Since the value of $s(\lambda, \theta_1, \theta_1', \theta_2, \theta_2')$ will flip between $+2$ and -2 as the hidden variable λ varies, we have

$$-2 \leq \int d\lambda \rho(\lambda)\, s(\lambda, \theta_1, \theta_1', \theta_2, \theta_2') \leq +2. \tag{7.40}$$

Defining

$$\begin{aligned} S_{\mathrm{HV}}(\theta_1, \theta_1', \theta_2, \theta_2') &= \int d\lambda \rho(\lambda)\, s(\lambda, \theta_1, \theta_1', \theta_2, \theta_2') \\ &= E_{\mathrm{HV}}\{\mathcal{A}(\theta_1)\mathcal{B}(\theta_2)\} - E_{\mathrm{HV}}\{\mathcal{A}(\theta_1)\mathcal{B}(\theta_2')\} \\ &\quad + E_{\mathrm{HV}}\{\mathcal{A}(\theta_1')\mathcal{B}(\theta_2)\} + E_{\mathrm{HV}}\{\mathcal{A}(\theta_1')\mathcal{B}(\theta_2')\}, \end{aligned} \tag{7.41}$$

we can write the ***Bell-Clauser-Horne-Shimony-Holt (BCHSH) inequalities***

$$-2 \leq S_{\mathrm{HV}}(\theta_1, \theta_1', \theta_2, \theta_2') \leq +2. \tag{7.42}$$

These are a generalization of the original Bell inequalities, which further assumes that $E_{\mathrm{HV}}\{\mathcal{A}(\theta_1)\mathcal{B}(\theta_1)\} = 1$. This is a very useful relation as it is based on the notion of classical hidden variables, although the result does not depend on the particular choice or nature of λ, and it lends itself well to experimental verification.

More precisely, if it was found that all experiments made on pairs of similarly-polarized photons were to obey equation (7.42), then the implication would be that classical models based on hidden variables could explain nature just as well as quantum mechanics. For someone like Einstein this would remove the notion that a polarization measurement made on a photon could determine the polarization state of the another photon on which a similar measurement is performed some very large distance away. This would also remove the possibility of faster-than-light information propagation and the lack of dependence of such spatially separated entities.

Alternatively, and perhaps more importantly, the BCHSH inequalities also provide a way to vindicate quantum mechanics. That is, if one could perform an experiment where these inequalities were violated, then it would negate the existence of hidden variables (and Einstein's view that quantum mechanics needs to be superseded by a more complete theory). For example, if we chose four orientations such that

$$\begin{aligned} \theta_1' &= \theta_2' + \frac{\pi}{8} && (7.43) \\ \theta_2 &= \theta_1' + \frac{\pi}{8} && (7.44) \\ \theta_1 &= \theta_2 + \frac{\pi}{8} \\ &= \theta_2' + \frac{3\pi}{8} && (7.45) \end{aligned}$$

we find from equation (7.28) that

$$S_{\mathrm{QM}}\left(\theta_1, \theta_1', \theta_2, \theta_2'\right) = 2\sqrt{2}, \tag{7.46}$$

which significantly exceeds the upper BCHSH inequality.

Since the enunciation of Bell's original and the BCHSH inequalities a large number of experiments have been performed that clearly shown repeated violation of equation (7.42), as predicted by quantum mechanics (through equation (7.46), for example). It follows that quantum mechanical entanglement is real and cannot be reproduced through classical concepts or theories, such as those based on hidden variables. As is clearly seen through the formalism of quantum theory, an entangled system (whether pairs of polarized photons, or electrons through their spins) cannot be treated independently and must be considered and analyzed as a compound system. This can be the case even if the entangled particles are separated by very large distances (some experiments on photon pairs were performed over a distance of several tens of kilometres; note however that there is no implication that "something" is travelling faster than the speed of light, nor is it needed). In other words, the characteristics of an entangled system cannot be described using the properties of its constituents taken separately.